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RESEARCH

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Oscillation for higher order differential equations with a middle term

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Dedicated to Professor I Kiguradze.

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Abstract

We study the existence of bounded oscillatory solutions for a higher order differential equation, considered as a perturbation of an associated linear equation. Jointly with this, we study the nonexistence of solutions vanishing at infinity and, as an application, we obtain in the linear case an asymptotic equivalency criterion.

1 Introduction

Consider the higher order nonlinear differential equation with a middle term

$$x^{(n)}(t) + q(t)x^{(n-2)}(t) + r(t)f(x(t)) = 0, \quad (1)$$

where $n \geq 3$ and $t \geq 0$. Throughout the paper, we assume that $f \in C(\mathbb{R})$ such that $f(u)u > 0$ for $u \neq 0$, $q \in C^1[0, \infty)$, $r \in C[0, \infty)$, and

$$q(t) \geq q_0 > 0, \quad \int_0^\infty |q'(t)| dt < \infty. \quad (H)$$

Hence, q is bounded from above and $\lim_{t \rightarrow \infty} q(t) = q_\infty < \infty$. Note that the function r may change sign.

By a solution of (1) we mean a function x differentiable up to order n which satisfies (1) on $[T_x, \infty)$, $T_x \geq 0$. A solution of (1) is said to be *proper* if $\sup\{|x(t)| : t \geq T\} > 0$ for any $T \geq T_x$. As usual, a solution x of (1) is said to be *oscillatory* if x changes sign for large t .

The aim of this paper is to study (1) as a perturbation of the linear differential equation

$$y^{(n)}(t) + q(t)y^{(n-2)}(t) = 0. \quad (2)$$

An important role in our approach is played by the relationship between solutions of (2) and those of the second order linear equation

$$h''(t) + q(t)h(t) = 0. \quad (3)$$

This approach is mainly motivated by previous results, obtained by Kiguradze in [1] for the special case $q(t) \equiv 1$, i.e., for the equation

$$x^{(n)}(t) + x^{(n-2)}(t) + r(t)f(x(t)) = 0. \quad (4)$$

It was shown in [1] that, if r is positive and sufficient large in some sense, then for n even every proper solution of (4) is oscillatory. Moreover, for n odd every proper solution of (4) is oscillatory, or is vanishing at infinity together with its derivatives, or admits the asymptotic representation

$$x(t) = c(1 + \sin(t - \varphi)) + \varepsilon(t),$$

where c, φ are suitable constants and ε is a continuous function for $t \geq 0$ which vanishes at infinity. According to [1], in this case (4) is said to have *property A'*.

The existence of solutions vanishing at infinity, and the closely related problem on existence of bounded oscillatory solutions, has attracted the attention in many papers, see, e.g. the monograph [2] and references therein. In particular, nonlinear equations with middle term have been investigated in many directions, as a perturbation of an associated linear equation. For third and fourth order equations, we refer especially to [3–5], in which the property A, or its generalizations, has been studied and to [6–10] for oscillation problems. In particular, in [5] a good and detailed discussion of known oscillation criteria is given as well.

The higher order equations of type (1) have been studied in [11–13]. More precisely, in [11, 12] the general equation

$$x^{(n)}(t) + \sum_{j=0}^{n-1} q_j(t)x^{(j)}(t) + r(t)|x(t)|^\lambda \operatorname{sgn} x(t) = 0, \quad \lambda > 1,$$

has been considered. In [11] sufficient conditions are obtained for the existence of solutions, which are equivalent to a polynomial. In [12] a criterion is given for existence of nonoscillatory solutions with non-zero limit at infinity. Moreover, for n even, an oscillation result is obtained too. In [13], (1) is studied as a perturbation of (2), under the assumption (H). Finally, the case of higher order equations with forcing term has been considered in the recent papers [14–16], see also the last section.

Observe that if q is a positive constant, then (2) has oscillatory and bounded solutions not vanishing at infinity. If q is not constant and (H) is satisfied, then these properties remain to hold for the second order equation (3), see, e.g. [17, Chapter 6], or [18, Theorem 2]. Thus, it is natural to ask under which assumptions these properties are valid also for (2) (when q is not constant) and for the more general case (1).

In this paper, we answer both these questions. In particular, our main result yields the existence of oscillatory solutions of (1), which are bounded and not vanishing at infinity. Our results complete recent ones in [13, Corollary 1, Corollary 3], extend similar ones in [1, Theorem 1.3, Theorem 1.4], which are proved for (4), and generalize ones in [3, 10], which are stated for the particular cases $n = 3, 4$.

In Section 3 we study the problem of the nonexistence of solutions vanishing at infinity of (1). This result will be employed in Section 4 and Section 5, namely to prove the existence of oscillatory solutions of (2) (Section 4) and the uniqueness of solutions of (1), which have the same asymptotics as solutions of (2) (Section 5).

We close the paper with an application that concerns the influence of the perturbing term r on the change of the oscillatory character, passing from (2) to the linear equation

$$x^{(n)}(t) + q(t)x^{(n-2)}(t) + r(t)x(t) = 0, \quad n \geq 3. \quad (5)$$

More precisely, we will give conditions under which (2) and (5) are asymptotically equivalent. Some suggestions for further researches complete the paper.

2 Preliminaries

We start with some basic properties of solutions of (3), which will be useful in the sequel. Obviously, assumption $q(t) \geq q_0 > 0$ implies that (3) is oscillatory. Moreover, the following holds.

Lemma 1 *Let (H) hold and let u be a nontrivial solution of (3). Then u is oscillatory and*

$$0 < \limsup_{t \rightarrow \infty} |u(t)| < \infty, \quad 0 < \limsup_{t \rightarrow \infty} |u'(t)| < \infty.$$

Proof Since q is of bounded variation for $t \geq 0$, all solutions of (3) are bounded together with their derivatives, see e.g. [18, Theorem 2]. Moreover, the function q can be represented as $q(t) = a(t)/b(t)$, where a, b are positive nonincreasing and differentiable functions such that $\lim_{t \rightarrow \infty} a(t) > 0$ and $\lim_{t \rightarrow \infty} b(t) > 0$, for details see [19, Lemma 5.4.1]. Hence, taking t_0 such that $u'(t_0) = 0$ and applying [13, Lemma 2], we get the assertion. \square

Let u, v be two linearly independent solutions of (3) with Wronskian equal to 1. Set

$$w(s, t) = u(s)v(t) - u(t)v(s), \quad \zeta(s, t) = \frac{\partial}{\partial t} w(s, t). \quad (6)$$

Then w and ζ are bounded in $[0, \infty) \times [0, \infty)$.

Equations (2) and (3) are strictly related. When $q(t) \equiv 1$, a basis of the space of solutions of (2) is given by

$$t^j, \quad j = 0, 1, \dots, n-3, \quad \sin t, \quad \cos t. \quad (7)$$

In the general case, that is, when q is not constant, it is easy to see that a basis of the space of solutions of (2) is given by

$$t^j, \quad j = 0, 1, \dots, n-3, \quad \Gamma_u, \quad \Gamma_v, \quad (8)$$

where

$$\Gamma_u = \int_0^t (t-s)^{n-3} u(s) ds, \quad \Gamma_v = \int_0^t (t-s)^{n-3} v(s) ds \quad (9)$$

and u, v are two independent solutions of (3). The question whenever the sets (7) and (8) are, roughly speaking, close as $t \rightarrow \infty$ is considered in Section 5.

We close this section by recalling the main result from [13], which plays a crucial role in our further consideration. Let

$$\bar{f}(u) = \max \{ |f(v)| : -u \leq v \leq u \}.$$

The symbol $g_1 = O(g_2)$ as $t \rightarrow \infty$ means, as usual, that there exists a constant M such that in a neighborhood of infinity

$$|g_1(t)| \leq M |g_2(t)|.$$

The following holds.

Theorem A ([13, Theorem 1]) Assume $n \geq 3$. Let for any positive constant μ and for some $j \in \{0, \dots, n-3\}$

$$\int_0^\infty t^{n-3} \tilde{f}(\mu t^j) |r(t)| dt < \infty. \quad (10)$$

Then, for any solution y of (2) such that $y(t) = O(t^j)$ as $t \rightarrow \infty$, there exists a solution x of (1) such that for large t

$$x^{(i)}(t) = y^{(i)}(t) + \varepsilon_i(t), \quad i = 0, \dots, n-1, \quad (11)$$

where ε_i are functions of bounded variation for large t and $\lim_{t \rightarrow \infty} \varepsilon_i(t) = 0$, $i = 0, \dots, n-1$.

Remark 1 If f satisfies in a neighborhood I of zero

$$|f(u)| \leq K|u| \quad (12)$$

for some $K > 0$, then every nontrivial solution of (1) is proper. This follows from [1, Theorem 11.5] with $h(t) = q_{\max} + K|r(t)|$ and $\omega(u) = u$ for $u \in I$.

3 Solutions vanishing at infinity

Our main result in this section is concerned with the nonexistence of solutions vanishing at infinity.

Theorem 1 Let $n \geq 3$, $f \in C^1(\mathbb{R})$ and

$$\int_0^\infty t^{n-3} |r(t)| dt < \infty. \quad (13)$$

Then (1) does not have nontrivial solutions x (oscillatory or nonoscillatory) satisfying

$$\lim_{t \rightarrow \infty} x(t) = 0. \quad (14)$$

To prove Theorem 1, some lemmas will be needed. The first result concerns the linear nonhomogeneous equation

$$z^{(n)}(t) + q(t)z^{(n-2)}(t) = F(t), \quad (15)$$

where $F \in C[0, \infty)$.

Lemma 2 Assume $n \geq 3$. Let $F \in C[0, \infty)$ satisfy

$$\int_0^\infty t^{n-3} |F(t)| dt < \infty. \quad (16)$$

Then

$$z(t) = (-1)^n \int_t^\infty \int_{\tau_{n-2}}^\infty \cdots \int_{\tau_1}^\infty w(s, \tau_1) F(s) ds d\tau_1 \cdots d\tau_{n-2},$$

where w is defined by (6), is a solution of (15). Moreover, there exists $M > 0$ such that for $t \geq 0$

$$|z(t)| \leq M \int_t^\infty s^{n-3} |F(s)| ds. \quad (17)$$

Proof If $F(t) \equiv 0$ for large t , then the assertion holds. Thus, assume that $F \not\equiv 0$ for large t .

First, we prove that z is well defined. Choose \bar{t} large so that

$$\frac{1}{q_0} \int_{\bar{t}}^\infty |q'(s)| ds = \ell_1 < 1. \quad (18)$$

Let m be a constant such that $|w(s, \tau)| \leq m$, $|\zeta(s, \tau)| \leq m$ for $s \geq \tau \geq \bar{t}$. From (16) we have

$$\begin{aligned} & \left| \int_{\bar{t}}^\infty \int_{\tau_{n-3}}^\infty \cdots \int_{\tau_1}^\infty w(s, \tau_1) F(s) ds d\tau_1 \cdots d\tau_{n-3} \right| \\ & \leq m \int_{\bar{t}}^\infty \cdots \int_{\tau_1}^\infty |F(s)| ds d\tau_1 \cdots d\tau_{n-3} \leq m \int_{\bar{t}}^\infty s^{n-3} |F(s)| ds < \infty. \end{aligned}$$

Hence, for fixed $T > \bar{t}$ the function

$$\alpha(t) = (-1)^n \int_t^T \int_{\tau_{n-2}}^\infty \cdots \int_{\tau_1}^\infty w(s, \tau_1) F(s) ds d\tau_1 \cdots d\tau_{n-2} \quad (19)$$

is well defined for $t \geq \bar{t}$. In particular, we have

$$\alpha^{(n-3)}(t) = \begin{cases} -\int_t^\infty \int_r^\infty F(s) w(s, r) ds dr & \text{if } n > 3, \\ -\int_t^T \int_r^\infty F(s) w(s, r) ds dr & \text{if } n = 3 \end{cases}$$

and for $n \geq 3$

$$\begin{aligned} \alpha^{(n-2)}(t) &= \int_t^\infty F(s) w(s, t) ds, \\ \alpha^{(n-1)}(t) &= \int_t^\infty F(s) \zeta(s, t) ds, \\ \alpha^{(n)}(t) &= -F(t) - q(t) \alpha^{(n-2)}(t). \end{aligned} \quad (20)$$

Consider the function

$$\gamma(t) = \alpha^{(n-1)}(t) + q(t) \alpha^{(n-3)}(t), \quad \bar{t} \leq t. \quad (21)$$

Then

$$\gamma'(t) = \alpha^{(n)}(t) + q(t) \alpha^{(n-2)}(t) + q'(t) \alpha^{(n-3)}(t) = -F(t) + q'(t) \alpha^{(n-3)}(t).$$

Integrating this equality on (t, T) , we get

$$\gamma(t) = \int_t^T F(s) ds - \int_t^T q'(s) \alpha^{(n-3)}(s) ds + \gamma(T).$$

From this, (20), and (21), we obtain

$$q(t)\alpha^{(n-3)}(t) = \int_t^T F(s) ds - \int_t^T q'(s)\alpha^{(n-3)}(s) ds - \int_t^\infty F(s)\zeta(s, t) ds + \gamma(T)$$

or, from (18),

$$|\alpha^{(n-3)}(t)| \leq \frac{1}{q_0} \left(\int_t^T |F(s)| ds + m \int_t^\infty |F(s)| ds + |\gamma(T)| \right) + \ell_1 \max_{t \leq s \leq T} |\alpha^{(n-3)}(s)|. \quad (22)$$

In view of (16), we have $|\alpha^{(n-3)}(T)| = 0$ or $|\alpha^{(n-3)}(T)| \leq m \int_T^\infty s|F(s)| ds$ according to $n = 3$ or $n > 3$, respectively. Moreover,

$$|\alpha^{(n-1)}(T)| \leq m \int_T^\infty |F(s)| ds \quad \text{for any } n \geq 3.$$

Hence, from (21) we obtain

$$\lim_{T \rightarrow \infty} |\gamma(T)| = 0, \quad n \geq 3. \quad (23)$$

For $n = 3$ we have $|\alpha(T)| = 0$ and

$$|\gamma(T)| = \left| \int_T^\infty F(s)\zeta(s, T) ds \right| \leq m \int_T^\infty |F(s)| ds. \quad (24)$$

From (22), in case $n > 3$ using (23) and letting $T \rightarrow \infty$, and in case $n = 3$ using (24) we get

$$|\alpha^{(n-3)}(t)| \leq \ell_2 \int_t^\infty |F(s)| ds + \ell_1 \sup_{s \geq t} |\alpha^{(n-3)}(s)|, \quad n \geq 3,$$

where $\ell_2 = (1 + 2m)/q_0$. Thus

$$\begin{aligned} |\alpha^{(n-3)}(t)| &\leq \sup_{\tau \geq t} |\alpha^{(n-3)}(\tau)| \\ &\leq \sup_{\tau \geq t} \left(\ell_2 \int_\tau^\infty |F(s)| ds + \ell_1 \sup_{s \geq \tau} |\alpha^{(n-3)}(s)| \right) \\ &\leq \ell_2 \int_t^\infty |F(s)| ds + \ell_1 \sup_{\tau \geq t} |\alpha^{(n-3)}(\tau)| \end{aligned}$$

or

$$(1 - \ell_1) \sup_{s \geq t} |\alpha^{(n-3)}(s)| \leq \ell_2 \int_t^\infty |F(s)| ds,$$

that is

$$|\alpha^{(n-3)}(t)| \leq \frac{\ell_2}{1 - \ell_1} \int_t^\infty |F(s)| ds. \quad (25)$$

When $n > 3$, since $\lim_{t \rightarrow \infty} |\alpha^{(i)}(t)| = 0$ for $i = 1, \dots, n - 3$, using (25) we get

$$|\alpha^{(i)}(t)| \leq \int_t^\infty |\alpha^{(i+1)}(s)| ds \leq \frac{\ell_2}{1 - \ell_1} \int_t^\infty \frac{(s - t)^{n-3-i}}{(n - 3 - i)!} |F(s)| ds.$$

From this, taking into account $\alpha(T) = 0$, we obtain

$$|\alpha(t)| = \left| \int_t^T \alpha'(s) ds \right| \leq M \int_t^\infty s^{n-3} |F(s)| ds, \quad (26)$$

where $M = \ell_2 / (1 - \ell_1)(n - 3)!$. If $n = 3$, then (26) holds by (25).

Now, for the sake of clarity, denote by $\alpha(t, T)$ the function α given in (19). Fix $\tilde{t} \geq \bar{t}$. In virtue of $\alpha(\tilde{t}, T) = \alpha(\tilde{t}, T_0) + \alpha(T_0, T)$ we have

$$|\alpha(\tilde{t}, T_1) - \alpha(\tilde{t}, T_2)| = |\alpha(T_0, T_1) - \alpha(T_0, T_2)|,$$

where $T_i \geq \tilde{t}$, $i = 0, 1, 2$. Thus for T_0 large so that

$$M \int_{T_0}^\infty s^{n-3} |F(s)| ds < \frac{\varepsilon}{2}$$

and from (26) we obtain

$$|\alpha(T_0, T_1) - \alpha(T_0, T_2)| \leq |\alpha(T_0, T_1)| + |\alpha(T_0, T_2)| < \varepsilon$$

for any $T_1 > T_0$, $T_2 > T_0$. Using the Cauchy criterion, in virtue of (26), there exists the finite limit

$$\lim_{T \rightarrow \infty} \alpha(\tilde{t}, T) = \alpha_\infty, \quad |\alpha_\infty| < \infty.$$

Hence, z is well defined. A direct computation shows that z is a solution of (15).

It remains to prove that z satisfies the estimation (17). Using (26) we have

$$|z(t)| = \lim_{T \rightarrow \infty} |\alpha(t)| \leq M \int_t^\infty s^{n-3} |F(s)| ds,$$

which yields (17). □

Remark 2 Under the stronger assumption $\int_0^\infty t^{n-2} |F(t)| dt < \infty$, Lemma 2 follows with a standard calculation.

Lemma 3 Let $n \geq 3$ and $y \in C^{(n-1)}[T, \infty)$, $T \geq 0$, be such that

$$\lim_{t \rightarrow \infty} y(t) = 0 \quad \text{and} \quad y^{(n-1)} \quad \text{is bounded on } [T, \infty). \quad (27)$$

Then

$$\lim_{t \rightarrow \infty} y^{(i)}(t) = 0 \quad \text{for } i = 0, 1, \dots, n-2.$$

Proof Let $b \in (T, \infty)$. Since y and $y^{(n-2)}$ are bounded, we can apply [2, Lemma 5.2] with $m = n - 1$ and we obtain

$$\max_{t \in [T, b]} |y^{(i)}(t)| \leq \frac{C_1}{(b - T)^i} \max_{t \in [T, b]} |y(t)| + C_2 \left(\max_{t \in [T, b]} |y(t)| \right)^{\frac{i}{n-1}} \left(\max_{t \in [T, b]} |y^{(n-1)}(t)| \right)^{\frac{n-1-i}{n-1}},$$

where $i = 1, 2, \dots, n-2$ and C_1, C_2 are suitable positive constants. Passing b to ∞ , there exists a positive constant C_3 such that

$$\sup_{t \in [T, \infty)} |y^{(i)}(t)| \leq C_3 \left(\sup_{t \in [T, \infty)} |y(t)| \right)^{\frac{i}{n-1}},$$

which gives the assertion, because $\lim_{t \rightarrow \infty} y(t) = 0$. \square

The next auxiliary result is a Gronwall type lemma, which proof is elementary and so it is omitted.

Lemma 4 *Let g and Ψ be nonnegative continuous functions for $t \geq t_0 \geq 0$ such that Ψ and Ψg belong to $L^1[t_0, \infty)$. If*

$$g(t) \leq A + \int_t^\infty \Psi(s)g(s)ds \quad (t \geq t_0)$$

for some nonnegative constant A , then

$$g(t) \leq A \exp\left(\int_t^\infty \Psi(s)ds\right) \quad (t \geq t_0).$$

In particular, if $A = 0$, then $g(t) = 0$ for each $t \in [t_0, \infty)$.

Proof of Theorem 1 Assume that there exists a nontrivial solution x of (1) defined on $[T_x, \infty)$ and satisfying (14).

Let u, v be linearly independent solutions of (3) with Wronskian 1. By Lemma 1,

$$u, \quad u', \quad v, \quad v' \quad \text{are bounded.} \quad (28)$$

Hence, $w(s, t)$, defined by (6), is bounded. Put $h(t) = x^{(n-2)}(t)$. Then h is a solution of the second order equation

$$h''(t) + q(t)h(t) = F(t), \quad (29)$$

where $F(t) = -r(t)f(x(t))$. Since $r \in L^1[0, \infty)$, from (14) and (28), we obtain

$$\int_{T_x}^\infty |u(s)F(s)|ds < \infty, \quad \int_{T_x}^\infty |v(s)F(s)|ds < \infty.$$

Taking into account (28), (29) and using the variation constant formula, we have

$$h(t) = x^{(n-2)}(t) = C_1 u(t) + C_2 v(t) - \int_t^\infty w(s, t)F(s)ds, \quad (30)$$

where C_1, C_2 are suitable constants.

Thus

$$x^{(n-1)}(t) = C_1 u'(t) + C_2 v'(t) - \int_t^\infty \zeta(s, t)F(s)ds. \quad (31)$$

From this and (28), we see that $x^{(n-1)}$ is bounded on $[T_x, \infty)$. By Lemma 3 we obtain

$$\lim_{t \rightarrow \infty} x^{(i)}(t) = 0, \quad i = 0, 1, \dots, n-2. \quad (32)$$

Hence, from (30)

$$\lim_{t \rightarrow \infty} [C_1 u(t) + C_2 v(t)] = 0.$$

We claim that $C_1 = C_2 = 0$. Indeed, if $C_1^2 + C_2^2 > 0$, then $C_1 u + C_2 v$ is nontrivial solution of (3) and applying Lemma 1 we get a contradiction.

Therefore, (30) implies

$$x^{(n-2)}(t) = - \int_t^\infty w(s, t) F(s) ds.$$

Integrating $n-2$ times and using (32), we get

$$x(t) = (-1)^{n-1} \int_t^\infty \int_{\tau_{n-2}}^\infty \cdots \int_{\tau_1}^\infty w(s, \tau_1) F(s) ds d\tau_1 \cdots d\tau_{n-2} \quad (33)$$

for $t \geq T_x$. Since $f \in C^1$, in view of the mean value theorem, there exists a function $\xi = \xi(s)$ such that

$$f(x(s)) = f'(\xi(s))x(s) \quad (34)$$

and $0 \leq |\xi(s)| \leq |x(s)|$. We have for $x(s) \neq 0$

$$f'(\xi(s)) = \frac{f(x(s))}{x(s)},$$

thus, because $f \in C^1$, the function $f'(\xi(s))$ is continuous and there exists $N > 0$ such that

$$|f'(\xi(s))| \leq N. \quad (35)$$

Thus, from (33), we obtain

$$\begin{aligned} |x(t)| &\leq \left| \int_t^\infty \int_{\tau_{n-2}}^\infty \cdots \int_{\tau_1}^\infty w(s, \tau_1) r(s) f(x(s)) ds d\tau_1 \cdots d\tau_{n-2} \right| \\ &= \left| \int_t^\infty \int_{\tau_{n-2}}^\infty \cdots \int_{\tau_1}^\infty w(s, \tau_1) r(s) f'(\xi(s)) x(s) ds d\tau_1 \cdots d\tau_{n-2} \right|. \end{aligned}$$

Hence, according to Lemma 2 with $F(t) = r(t)f'(\xi(t))x(t)$ we get for large t

$$|x(t)| \leq MN \int_t^\infty s^{n-3} |r(s)| |x(s)| ds.$$

By Lemma 4, we find that x is identically zero for large t . Since $f \in C^1$, (34) and (35) imply (12), by Remark 1 every nontrivial solution x of (1) is proper and so x is identically zero for $t \geq T_x$. The proof is complete. \square

Remark 3 If $f(u) = |u|^\lambda \operatorname{sgn} u$, $\lambda \geq 1$, then the assumption $f \in C^1(\mathbb{R})$ in Theorem 1 is satisfied. If $\lambda < 1$, then $f \notin C^1(\mathbb{R})$ and Theorem 1 is not valid. The following example illustrates that assumptions of Theorem 1 are optimal.

Example 1 Consider the third order equation

$$x'''(t) + x'(t) + r(t)|x(t)|^\lambda \operatorname{sgn} x(t) = 0, \quad t \geq 0, \quad (36)$$

where $r(t) = \alpha(\alpha + 1)(\alpha + 2)(t + 1)^{\alpha\lambda - \alpha - 3} + \alpha(t + 1)^{\alpha\lambda - \alpha - 1}$ and $\alpha > 0$.

If $\lambda < 1$, then $\int_0^\infty r(t) dt < \infty$, i.e. condition (13) is satisfied, however, we have $f \notin C^1(\mathbb{R})$. If $\lambda \geq 1$, condition (13) is not satisfied.

One can check that in both cases $x(t) = (t + 1)^{-\alpha}$ is a solution vanishing at infinity of (36). This shows the strictness of both assumptions of Theorem 1, that is, $f \in C^1(\mathbb{R})$ and (13).

4 Oscillation in the linear case

In this section we prove the existence of oscillatory solutions of (2), which are bounded and not vanishing at infinity.

Theorem 2 Let $n \geq 3$, u be a nontrivial solution of (3) and

$$\int_0^\infty s^{n-3} |q'(s)| ds < \infty. \quad (37)$$

Then (2) has an oscillatory solution ϕ such that

$$\phi(t) = \begin{cases} u'(t) + \varepsilon(t) & \text{for } n \text{ odd,} \\ u(t) + \varepsilon(t) & \text{for } n \text{ even,} \end{cases}$$

where ε is a continuous function on $[0, \infty)$ and $\lim_{t \rightarrow \infty} \varepsilon(t) = 0$. In particular,

$$0 < \limsup_{t \rightarrow \infty} |\phi(t)| < \infty.$$

To prove this theorem, we give asymptotic expressions of the integrals in (8).

Lemma 5 Let $n \geq 3$ and (37) hold. If u is a nontrivial (oscillatory) solution of (3), then there exist constants c_i , $i = 0, 1, \dots, n - 2$, $c_{n-2} \neq 0$, and a function ε such that

$$\Gamma_u(t) = \begin{cases} \sum_{i=0}^{n-3} c_i t^i + c_{n-2} u'(t) + \varepsilon(t) & \text{for } n \text{ odd,} \\ \sum_{i=0}^{n-3} c_i t^i + c_{n-2} u(t) + \varepsilon(t) & \text{for } n \text{ even,} \end{cases}$$

where ε is a continuous function on $[0, \infty)$ and $\lim_{t \rightarrow \infty} \varepsilon(t) = 0$.

Proof In view of (H), we have $\lim_{t \rightarrow \infty} q(t) = q_\infty$, $0 < q_\infty < \infty$. Let μ be an integer, $\mu \geq 1$. Using (3) we get

$$\begin{aligned} \int_0^t \frac{u(s)}{q^{\mu-1}(s)} ds &= - \int_0^t \frac{u''(s)}{q^\mu(s)} ds = - \frac{u'(t)}{q^\mu(t)} + \frac{u'(0)}{q^\mu(0)} - \mu \int_0^t \frac{q'(s)u'(s)}{q^\mu(s)} ds \\ &= \alpha - (q_\infty)^{-\mu} u'(t) + \varepsilon_0(t), \end{aligned} \quad (38)$$

where

$$\alpha = \frac{u'(0)}{q^\mu(0)} - \mu \int_0^\infty \frac{q'(s)u'(s)}{q^\mu(s)} ds,$$

$$\varepsilon_0(t) = \mu \int_t^\infty \frac{q'(s)}{q^\mu(s)} ds + ((q_\infty)^{-\mu} - (q(t))^{-\mu})u'(t)$$

and $\lim_{t \rightarrow \infty} \varepsilon_0(t) = 0$.

Similarly, let ν be an integer, $\nu \geq 1$. Using (3) we have

$$\begin{aligned} \int_0^t \frac{(t-s)u(s)}{q^{\nu-1}(s)} ds &= - \int_0^t \frac{(t-s)u''(s)}{q^\nu(s)} ds \\ &= \frac{u'(0)}{q^\nu(0)}t - \frac{u(t)}{q^\nu(t)} + \frac{u(0)}{q^\nu(0)} - \nu \int_0^t \frac{q'(s)u(s)}{q^{\nu+1}(s)} ds \\ &\quad - \nu \int_0^t \frac{(t-s)q'(s)u'(s)}{q^{\nu+1}(s)} ds \\ &= \beta t + \gamma - (q_\infty)^{-\nu}u(t) + \varepsilon_1(t), \end{aligned} \quad (39)$$

where

$$\begin{aligned} \beta &= \frac{u'(0)}{q^\nu(0)} - \nu \int_0^\infty \frac{q'(s)u'(s)}{q^{\nu+1}(s)} ds, \\ \gamma &= \frac{u(0)}{q^\nu(0)} + \nu \int_0^\infty \frac{q'(s)su'(s)}{q^{\nu+1}(s)} ds - \nu \int_0^\infty \frac{q'(s)u(s)}{q^{\nu+1}(s)} ds, \\ \varepsilon_1(t) &= \nu \int_t^\infty \frac{q'(s)u(s)}{q^{\nu+1}(s)} ds + \nu \int_t^\infty \frac{q'(s)(t-s)u'(s)}{q^{\nu+1}(s)} ds + ((q_\infty)^{-\nu} - (q(t))^{-\nu})u(t) \end{aligned}$$

and $\lim_{t \rightarrow \infty} \varepsilon_1(t) = 0$.

When $n = 3$, the assertion follows from (38) with $\mu = 1$. Similarly, when $n = 4$, the assertion follows from (39) with $\nu = 1$.

Now, let $n \geq 5$. Let j be an integer and $n - 3 \geq j \geq 2$. Fixed $k > 0$ and integrating by parts, we obtain

$$\begin{aligned} \int_0^t \frac{(t-s)^j}{q^k(s)} u(s) ds &= - \int_0^t \frac{(t-s)^j u''(s)}{q^{k+1}(s)} ds \\ &= \frac{u'(0)}{q^{k+1}(0)} t^j - j \int_0^t \frac{(t-s)^{j-1} u'(s)}{q^{k+1}(s)} ds - (k+1) \int_0^t \frac{(t-s)^j q'(s) u'(s)}{q^{k+2}(s)} ds. \end{aligned}$$

Similarly

$$\int_0^t \frac{(t-s)^{j-1} u'(s)}{q^{k+1}(s)} ds = \frac{-u(0)}{q^{k+1}(0)} t^{j-1} + (j-1) \int_0^t \frac{(t-s)^{j-2} u(s)}{q^{k+1}(s)} ds.$$

Applying Lemma 1 and (37), we have

$$\begin{aligned} \int_0^\infty \frac{(t-s)^{j-1} q'(s) u(s)}{q^{k+2}(s)} ds &= \sum_{m=0}^{j-1} d_m t^m, \\ \int_0^\infty \frac{(t-s)^j q'(s) u'(s)}{q^{k+2}(s)} ds &= \sum_{m=0}^j \bar{d}_m t^m, \end{aligned}$$

where

$$d_m = (-1)^{j-m-1} \binom{j-1}{m} \int_0^\infty \frac{q'(s)}{q^{k+2}(s)} s^{j-m-1} u(s) ds,$$

$$\bar{d}_m = (-1)^{j-m} \binom{j}{m} \int_0^\infty \frac{q'(s)}{q^{k+2}(s)} s^{j-m} u'(s) ds.$$

Hence

$$\int_0^t \frac{(t-s)^j u(s)}{q^k(s)} ds = \sum_{m=0}^j D_m t^m - j(j-1) \int_0^t \frac{(t-s)^{j-2} u(s)}{q^{k+1}(s)} ds + H_j(t), \quad (40)$$

where D_j are constants and

$$H_j(t) = (k+1) \int_t^\infty \frac{(t-s)^{j-1} q'(s)}{q^{k+2}(s)} (ju(s) + (t-s)u'(s)) ds.$$

Since

$$|H_j(t)| \leq (k+1) \int_t^\infty \frac{s^{j-1} |q'(s)|}{q^{k+2}(s)} (j|u(s)| + s|u'(s)|) ds,$$

in virtue of Lemma 1 and (37), we get $\lim_{t \rightarrow \infty} H_j(t) = 0$.

Let n be odd, $n \geq 5$. By using recursively (40) with

$$j = \{2, 4, 6, \dots, n-3\} \quad \text{and} \quad k = (n-3-j)/2,$$

we obtain the following estimation for the function Γ_u given by (9):

$$\Gamma_u(t) = \sum_{i=0}^{n-3} c_i t^i - (n-3)! \int_0^t \frac{u(s)}{q^{(n-3)/2}(s)} ds + K_1(t), \quad (41)$$

where c_i are suitable constants and K_1 is a continuous function on $[0, \infty)$ and $\lim_{t \rightarrow \infty} K_1(t) = 0$.

Choosing $\mu = (n+1)/2$ in (38), from (41) the assertion follows for n odd, $n \geq 5$.

Finally, let n be even, $n \geq 6$. By using a similar argument to the one above given and applying recursively (40) with

$$j = \{3, 5, 7, \dots, n-3\} \quad \text{and} \quad k = (n-3-j)/2,$$

we obtain the following estimation for the function Γ_u :

$$\Gamma_u(t) = \sum_{i=0}^{n-3} \bar{c}_i t^i - (n-3)! \int_0^t \frac{(t-s)u(s)}{q^{n/2-2}(s)} ds + K_2(t), \quad (42)$$

where \bar{c}_i are suitable constants and K_2 is a continuous function on $[0, \infty)$ and $\lim_{t \rightarrow \infty} K_2(t) = 0$.

Choosing $\nu = (n-2)/2$ in (39), from (42) the assertion follows. \square

Proof of Theorem 2 By Lemma 5, (2) has a solution $u'(t) + \varepsilon(t)$ if n is odd, and $u(t) + \varepsilon(t)$ if n is even, where $\lim_{t \rightarrow \infty} \varepsilon(t) = 0$. According to Lemma 1 this solution is oscillatory, bounded, and not vanishing at infinity. \square

5 Oscillation in the nonlinear case

Our main results are given by the following.

Theorem 3 Let $n \geq 3$ and u be a nontrivial solution of (3).

Assume (37) and

$$\int_0^\infty t^{n-3} |r(t)| dt < \infty. \quad (43)$$

Then for any real numbers c_0, c_1 , (1) has a solution x , defined on $[T_x, \infty)$, $T_x \geq 0$, such that

$$x(t) = \begin{cases} c_0 + c_1 u'(t) + \varepsilon(t) & \text{for } n \text{ odd,} \\ c_0 + c_1 u(t) + \varepsilon(t) & \text{for } n \text{ even,} \end{cases}$$

where ε is a continuous function on $[T_x, \infty)$ and $\lim_{t \rightarrow \infty} \varepsilon(t) = 0$.

Consequently, (1) has oscillatory solutions x such that

$$0 < \limsup_{t \rightarrow \infty} |x(t)| < \infty.$$

Proof By Theorem 2, (2) has an oscillatory solution ϕ , which is bounded and not vanishing at infinity. Applying Theorem A with $j = 0$, (1) has a solution with the same asymptotic properties as that one of (2). \square

Theorem 4 Let $n \geq 3$ and u, v be two linearly independent solutions of (3). Assume (37) and for any positive constant μ

$$\int_0^\infty t^{n-3} \bar{f}(\mu t^{n-3}) |r(t)| dt < \infty, \quad (44)$$

where the function \bar{f} is defined in the Preliminaries.

Then for any vector $(c_0, c_1, \dots, c_{n-1}) \in \mathbb{R}^n$ there exists a solution x of (1), defined on $[T_x, \infty)$, $T_x \geq 0$, such that

$$x(t) = \begin{cases} \sum_{i=0}^{n-3} c_i t^i + c_{n-2} u'(t) + c_{n-1} v'(t) + \varepsilon(t) & \text{for } n \text{ odd,} \\ \sum_{i=0}^{n-3} c_i t^i + c_{n-2} u(t) + c_{n-1} v(t) + \varepsilon(t) & \text{for } n \text{ even,} \end{cases} \quad (45)$$

where ε is a continuous function on $[T_x, \infty)$ and $\lim_{t \rightarrow \infty} \varepsilon(t) = 0$.

If, in addition, $f \in C^1(\mathbb{R})$ and there exists $M > 0$ such that

$$|f'(u)| \leq M \bar{f}(u) \quad \text{for large } |u|, \quad (46)$$

then the solution x given by (45) is unique.

Proof Existence. As noticed above, the set (8) is a basis for the space of solutions of (2). Applying Theorem A with $j = n - 3$ and Theorem 3, (1) has a solution x satisfying (45).

Uniqueness. For the sake of simplicity, let n be even. The case n odd follows in a similar way. Suppose, by contradiction, that for $(c_0, c_1, \dots, c_{n-1}) \in \mathbb{R}^n$ there exist two different solutions x and \bar{x} of (1) satisfying (45). Then for $\bar{z}(t) = x(t) - \bar{x}(t)$ we have

$$\lim_{t \rightarrow \infty} \bar{z}(t) = 0 \quad (47)$$

and

$$\bar{z}^{(n)}(t) + q(t)\bar{z}^{(n-2)}(t) + r(t)(f(x(t)) - f(\bar{x}(t))) = 0.$$

In view of the mean value theorem, there exists a function $\xi = \xi(s)$ such that

$$f(x(t)) - f(\bar{x}(t)) = f'(\xi(t))(x(t) - \bar{x}(t))$$

and

$$\min\{x(t), \bar{x}(t)\} \leq \xi(t) \leq \max\{x(t), \bar{x}(t)\}.$$

Then we have

$$|\xi(t)| \leq M_1 t^{n-3}.$$

Therefore \bar{z} is a solution of

$$z^{(n)}(t) + q(t)z^{(n-2)}(t) + R(t)z(t) = 0, \quad (48)$$

where $R(t) = r(t)f'(\xi(t))$, for $t \geq 1$ and a suitable constant $M_1 > 0$.

Let (46) hold for $|u| \geq u_0 > 0$. Then, in view of (44), the condition (13) for (48) is satisfied, because

$$\int_0^\infty t^{n-3} |R(t)| dt \leq \int_0^\infty t^{n-3} |r(t)| (M_2 + M\bar{f}(M_1 t^{n-3})) dt < \infty,$$

where $M_2 = \max_{|v| \leq u_0} |f'(v)|$. Hence we can apply Theorem 1 to (48), which gives a contradiction with (47). \square

Remark 4 Theorem 3 extends [1, Theorem 1.4] and Theorem 4 [1, Theorem 1.3] stated for (1) with $r(t) > 0$.

An application of Theorems 3 and 4 is the following. Consider the Emden-Fowler type equation

$$x^{(n)}(t) + x^{(n-2)}(t) + r(t)|x(t)|^\lambda \operatorname{sgn} x(t) = 0, \quad (49)$$

where $\lambda > 0$. Then (46) is satisfied for any $\lambda > 0$ and (44) reads

$$\int_0^\infty t^{(n-3)(\lambda+1)} |r(t)| dt < \infty. \quad (50)$$

Thus, according to Theorem 4, for a fixed vector $(c_0, c_1, \dots, c_{n-1})$ there exists a unique solution of (49) which has the asymptotic representation (45).

Now, consider (49), where n is even, $r(t) > 0$ for $t \geq 0$ and $\lambda \neq 1$. By [1, Corollary 1.6], if $\int_0^\infty t^{n-3} r(t) dt = \infty$, then every solution of (49) is oscillatory. Obviously, condition (37) is satisfied. Therefore, the condition (43) in Theorem 3 is optimal.

From this and Theorem 3 we get the following.

Corollary 1 *Let $\lambda > 1$, n even, $n \geq 4$ and $r(t) > 0$ for $t \geq 0$. Then (49) has oscillatory solutions.*

6 Asymptotic equivalence of linear equations

In this section we present another consequence of our results.

Consider the linear equation

$$x^{(n)}(t) + q(t)x^{(n-2)}(t) + r(t)x(t) = 0, \quad n \geq 3, \quad (5)$$

and denote by S_x and S_y the solution space of (5) and (2), respectively.

We say that (2) and (5) are *asymptotically equivalent*, if there exists a map $T: S_y \rightarrow S_x$ such that for every $y \in S_y$ there exists a unique $x \in S_x$ such that $T(y) = x$ and

$$\lim_{t \rightarrow \infty} (x(t) - y(t)) = 0.$$

Applying Theorem A and Theorem 1 we get the following.

Theorem 5 *Assume $n \geq 3$ and*

$$\int_0^\infty t^{2n-6} |r(t)| dt < \infty. \quad (51)$$

Then (2) and (5) are asymptotically equivalent.

Proof As we noticed above, functions t^j , Γ_u and Γ_v are linearly independent solutions of (2).

By Theorem A, there exist functions η_j , $j = 0, \dots, n-1$, which tend to zero as $t \rightarrow \infty$, such that

$$x_j(t) = t^j + \eta_j(t), \quad j = 0, \dots, n-3,$$

$$x_{n-2}(t) = \Gamma_u(t) + \eta_{n-2}(t),$$

$$x_{n-1}(t) = \Gamma_v(t) + \eta_{n-1}(t)$$

are solutions of (5). Hence, applying again Theorem A, we get

$$\lim_{t \rightarrow \infty} \eta_j^{(i)}(t) = 0, \quad i = 1, \dots, n-1. \quad (52)$$

We show that η_j are uniquely determined. Without loss of generality, assume by contradiction that there exist η_0 and $\bar{\eta}_0$ such that $\eta_0 \not\equiv \bar{\eta}_0$ and

$$x_0(t) = 1 + \eta_0(t), \quad \bar{x}_0(t) = 1 + \bar{\eta}_0(t)$$

are solutions of (5). Then $\bar{x}_0 - x_0$ is also a solution of (5) and tends to zero as $t \rightarrow \infty$. This is a contradiction with Theorem 1. For $j \in \{1, \dots, n-1\}$ we proceed by the same way.

A standard argument shows that solutions x_j , $j = 0, \dots, n-1$, are linearly independent. Set

$$z(t) = \sum_{j=0}^{n-1} c_j x_j(t) \quad (53)$$

and let us prove that $z(t) \equiv 0$ implies that $(c_0, c_1, \dots, c_{n-1})$ is the zero vector in \mathbb{R}^n . From (53) we obtain

$$0 \equiv \frac{d^{n-2}}{dt^{n-2}} z(t) = c_{n-2} u(t) + c_{n-1} v(t) + c_{n-2} \eta_{n-2}^{(n-2)}(t) + c_{n-1} \eta_{n-1}^{(n-2)}(t).$$

From this, (52) with $i = n-2$ and Lemma 1, we get $c_{n-2} = c_{n-1} = 0$. Therefore,

$$z(t) = \sum_{j=0}^{n-3} c_j x_j(t).$$

Since

$$0 \equiv \frac{d^{n-3}}{dt^{n-3}} z(t) = c_{n-3} + \eta_{n-3}^{(n-3)}(t)$$

and $\lim_{t \rightarrow \infty} \eta_{n-3}^{(n-3)}(t) = 0$, we have $c_{n-3} = 0$. Proceeding in the same way we get $c_i = 0$ for $i = 0, \dots, n-2$.

Denote

$$\begin{aligned} \bar{S}_x &= \{x_j(t), j = 0, \dots, n-1\}, \\ \bar{S}_y &= \{t^j, j = 0, \dots, n-3, \Gamma_u(t), \Gamma_v(t)\}. \end{aligned}$$

Since η_j are uniquely determined, there exists a 1-1 map between the sets \bar{S}_x and \bar{S}_y and so the assertion follows. \square

Remark 5 The assumption (37) is not needed in Theorem 5.

7 Open problems

- (1) Does Corollary 1 hold for n even and $\lambda \leq 1$?
- (2) Let q be bounded, but not of bounded variation on $[0, \infty)$. Then (3) can have unbounded oscillatory solutions, see, e.g., [17, Chapter VI-5, Theorem 3]. Similarly, if (3) is oscillatory and $\lim_{t \rightarrow \infty} q(t) = 0$, then (3) can have again unbounded oscillatory solutions, as the Euler equation illustrates. In these cases, it should be interesting to find conditions under which (1) has unbounded oscillatory solutions too.
- (3) Let q be unbounded. Thus, as the Armellini-Tonelli-Sansone theorem shows, (3) can have (oscillatory) solutions which tend to zero as $t \rightarrow \infty$, see [20] for a detailed survey on this topic. In particular, in [20, 21] the boundedness and the existence of vanishing at infinity solutions are investigated for second order linear equations with advanced

arguments. As before, also in this situation, it should be an interesting problem studying asymptotic properties of possible oscillatory solutions of (1). Note that for (1) with $n = 4$, conditions which ensure that all solutions are oscillatory, can be found in [6, 7] or in [8, Theorem 4].

- (4) Let g be a continuous function on \mathbb{R} such that

$$\lim_{|u| \rightarrow \infty} \int_0^u g(\sigma) d\sigma = \infty.$$

Under assumptions (H), any solution of the nonlinear equation

$$y''(t) + q(t)g(y(t)) = 0$$

is bounded together with its first derivative, see, e.g., [18, Theorem 4]. Thus, motivated by the results here obtained, we can ask under which conditions the nonlinear equation

$$x^{(n)}(t) + q(t)g(x^{(n-2)}(t)) + r(t)f(x(t)) = 0$$

has oscillatory bounded solutions.

- (5) Recently, oscillation of equations with a forcing term have been studied. For example, the boundedness of any solutions is studied in [15] and the periodic case in [14]. Moreover, in [16] a two-term equation with forcing term e has been considered and the oscillation is studied under additional conditions on the function r . Thus, it seems interesting to extend our study to the existence of oscillatory solutions and solutions vanishing at infinity for the equation with the forcing term

$$x^{(n)}(t) + q(t)x^{(n-2)}(t) + r(t)f(x(t)) = e(t).$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the manuscript and read and approved the final draft.

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